



From BASEL III to BASEL IV and beyond: Expected shortfall and expectile risk measures

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ABSTRACT

The article contributes to the ongoing search for a market risk measure that is both coherent and elicitable. We compare two traditional measures, namely Value-at-Risk and the expected shortfall, with another relatively novel one established on the expectile probability term. Our research is based on five models: Black–Scholes, exponential tempered stable, Heston, Bates and another stochastic volatility model with a tempered stable jump correction. We apply the general Fourier inversion formula to derive closed form formulas for calculating not only the expectile based risk measure but also the Value-at-Risk and the expected shortfall. These models are calibrated by combining nonlinear programming with simulated annealing at a moving window. Additionally, we compare the generated values of the risk measures with the real ones. Last but not least, we modify the expectile based risk measure as well as the expected shortfall by introducing correction coefficients.

1. Introduction

For the business it is of utmost importance the way it measures the financial risks. This is more than true for supervised entities, like the credit institutions. Banks must obey the requirements of the national regulators. The national bank supervisor adopts the acts of the Basel Committee on Banking Supervision (BCBS) and adjusts their implementation to the regulated entities, if this is allowed. Hence, for the banks around the globe matters the risk measurement approach recommended by the BCBS. BCBS publishes obligatory recommendations how to measure the market risk. The evolution of the risk measures is marked with the following milestones. By promulgating amendments to the so-called Basel Capital Accord (aka Basel II) in 1996 the BCBS codified the business practice to apply the Value-at-Risk (VaR, hereafter) – see p. 44 of BCBS, 1996.¹ Later, on 2012 the expected shortfall (ES) replaced the VaR for calculating the capital requirements via Basel III due to identified weaknesses of the latter – see BCBS, 2013, p.3.² The aptitude of the risk measure to be forecasted is important per se (Dai et al., 2021, Hallin & Trucíos, 2021) and for backtesting purposes – Argyropoulos and Panopoulou (2019), Lazar and Xue (2020), Del Brio et al. (2020), Dimitriadis and Schnaitmann (2021). Also what matters

for selecting a risk measure is the costs of estimating the measure – simple, tractable and transparent estimation are the preferred choice factors, see for example Catania and Luati (2021).

Robust risk measures are at the center of business attention to correctly estimate the capital reserves (Righi et al., 2020); hence, the asset type (Ahelegbey et al., 2021) and potential mispricing (Yang & Ma, 2021) have impact on selecting the risk measure. Despite the progress in describing the risk market phenomena, not considered were the possibilities to remedy the VaR subadditivity, see Chen (2018), nor the arguments in favor of the ES aptitude to backtesting – for some arguments see Roccioletti (2016) pp. 58–59. There are authors, see Section 7.5 of Rachev et al. (2008), who argue that it is possible to backtest the ES, if a certain structure of the log-return tails is assumed. For example Bianchi et al. (2019) p. 266–270, apply the tempered stable distribution to the ES calculation, whereas Song and Li (2022) uses the normal inverse Gaussian distribution for predicting VaR. The non-elicitability of the ES necessitates the application of complicated approaches, like two steps VaR-based algorithm (Storti & Wang, 2022) or to forecast the VaR-ES pair (Gao et al., 2022), see also Burzoni et al. (2022) for some other generalizations. On the other hand, recently

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¹ <https://www.bis.org/publ/bcbs24.pdf>

² <https://www.bis.org/publ/bcbs265.pdf>

published articles (Ziegel, 2016 and Bellini and DiBernardino (2017)) mention the VaR incoherence and the ES non-elicitability as arguments to introduce the expectile based risk measure (ERM, hereafter). A benefactor for its selecting for the next risk measure is the clear economic interpretation of the expectile, see for example Kuan et al. (2009). The related *gain-loss* and *omega ratios* are considered in Chen (2018). An innovation of adopting the ERM is the application of new ratios to measure the portfolio performance, see for example Lin et al. (2021), Avci and Avci (2021), Hu and Zheng (2020) and Bellini et al. (2021). The reinsurance business uses the expectile measure under the name *optimal retention*, see Cai and Tan (2007), Gerber et al. (2019), and Indah and Fadilla (2020). The need to capture the stylized facts of return tails entails attempts to apply heavy tails methods – see Daouia et al. (2021), Hu et al. (2021), Liu and Wang (2021), Opschoor and Lucas (2021), Davison et al. (2022). Alternatively, Mihoci et al. (2021) use mixing Gaussian and Laplace densities to calculate the ERM and explore its sensitivity and robustness. Tadese and Drapeau (2020) demonstrate how to characterize the expectile via the ES, and to derive the lower and upper bounds of the expectile in terms of the ES. Bellini et al. (2022) compare time series of the S&P 500 index through inter-expected shortfall differences and inter-expectile differences. Based on the VaR, ES, and ERM, Marcin and Schmidt (2022) introduce an efficient way to reduce the backtesting bias in the case of heavy-tailed and heteroscedastic data. On the other hand, the expectile might become sensitive to the magnitude of extreme losses by the introduction of the conditional ERM – see Ren et al. (2022).

Since the new risk measure has to compensate the drawbacks of its ancestors, our research aims to contribute to the ongoing quest for a better choice. It treats the ERM as the only risk measure that is both coherent and elicitable. Alternatively to the establish practice in the mentioned above sources, we approach the risk measurement problem in a different manner. We assume that the considered object, particularly the S&P500 index, is driven by a stochastic process instead to investigate its statistical properties directly from the available empirical data. Once we chose the particular stochastic process for our purposes, we may calibrate it to the historical data. There are several important advantages of this approach. First, in this way we can capture some inherent features like fat-tailed and asymmetric distributions, frequent sudden jump movements, leverage effect, volatility clustering, long range dependence, etc., which, directly or indirectly, influence the risk characteristics. On the other hand, the risk is mainly affected on the extreme events, i.e. the tail observations. They are not so much and often it is hard to be distinguished from the so-called statistical errors. This way the stochastic process we use fulfills the missing observations as at the same time it filters the empirical data from statistical errors. Another interesting advantage our method exhibits is the possibility to incorporate an additional information from the real markets. The standard approach for risk measuring of some asset is based only on its historical movements. But many hidden information is available at the derivative markets. Our approach allows extracting of this information by a joint calibration to the historical data as well as to the available derivatives – see Zaeviski et al. (2014). It is well known that differently to the main asset, whose behavior is measured by the real-life probability measure, the derivative prices obey to the so called risk-neutral measure. These two measures are related to each other by some non-changeable parameters – for example in the Black–Scholes model this is the volatility. This way, based on both of the asset history and its derivatives' prices, we can describe more realistically the asset and thus more precisely to measure its risk. However, this aspect is not the subject of the present work and is subject to a further research. Another important novelty of our research is the method we suggest for shifting to a new risk measure. It is of an outstanding interest for the financial institutions this to be made as smoothly as possible to avoid a substantially increase of the required capital reserves. It is currently accepted that this is made by a change of the quantiles for the different measures. Based on the standard normal

distribution and many historical observations, BCBS enforces the 2.5-th percentile of the ES to succeeded the 1-st percentile of VaR, see BCBS 2013, p. 18.³ Later Li and Wang (2022) investigate the probability equivalent level that allows to shift among several risk measures. In the same vein Bellini and DiBernardino (2017) propose the 0.01 VaR and the 0.25 ES to be accompanied by the 0.00145 expectile. Also, Chen (2018) provides further VaR-ES-expectile relations. Unfortunately, this assumption can be seriously criticized due to the strongly non-Gaussian structure observed at all financial markets. Another shortcoming of this approach is that the quantile level change leads to adding or losing some tail/risk information. To avoid these disadvantages we suggest the use of correction coefficients between the measures – the values of the ES and the ERM are taken not itself, but multiplied by some coefficients. This way the different measures generate relatively similar values preserving the whole tail information. At last but not least the approach we suggest allows a large flexibility and changeability during the time.

We base our approach on five major models which we use to describe the S&P500 index namely the famous Black and Scholes (1973) model, exponential tempered stable model and the models of Heston (1993), Bates (1996), and Zaeviski et al. (2014). The first two of them are with stationary and independent increments – Gaussian and Lévy style, respectively. Differently to the Gaussian assumption, tempered stable one allows an asymmetry, jumps, and a more mass in the tails. The rest of the models exhibit stochastic volatility – this way we can capture the phenomena of leverage effect, volatility clustering, and long range dependence. In addition the models of Bates (1996) and Zaeviski et al. (2014) combine all mentioned above market characteristics adding jumps in the Heston framework – the first one by a compound Poisson process with normally distributed innovations, whereas the second one uses infinite activity Lévy processes, particularly tempered stable. A main feature of these models, except the Black and Scholes (1973) one, is that they do not have probability density functions in a closed form – only the characteristic functions are available. This leads to the necessity of some Fourier inversion methods.⁴

Our results can be summarized in several directions. First, we derive closed form formulas for the risk measures (VaR, ES, and ERM) based on the mentioned above stochastic processes. They are based on the general Fourier inversion formula. Second, we calibrate them moving a window with 1000 observations through the empirical data. Then we generate the corresponding values of the risk measures and compare them to the real ones. By the way, the method for extracting these measures from the empirical data has its own significance. The calibration method we present is important itself. It is well-known that finding the minimum of a multi-parameter function is a challenging task. We suggest to combine the use of nonlinear programming with simulated annealing. After we fit the models, we provide many numerical experiments to compare the pros and cons of different models for back-testing, forecasts, goodness of fits, etc. in the light of risk measurement. Based on the historical sample we use, we suggest the value of 0.6 for the mentioned above ES to VaR correction coefficient and 1.25 for the ERM to VaR one.

The paper is organized as follows. The essence of the VaR, ES, and ERM as well as the way we derive their empirical values are presented in Section 2. The closed form formulas for VaR, ES, and ERM via the characteristic functions are obtained in Section 3. We present our calibration methodology and the numerical results it leads to in Section 4. The mathematical proofs of the risk measures' formulas are provided in Appendix.

³ <https://www.bis.org/publ/bcbs265.pdf>

⁴ Some similar approaches can be found in Kim et al. (2010), Bormetti et al. (2010), Rachev et al. (2011), and Nguyen and Nguyen (2017).

2. Risk measures

We present in this section the definitions and the main properties of three risk measures we shall use later. The first two of them are the classical value-at-risk (VaR) and the expected shortfall (ES) – in fact the ES is known previously as the average-value-at-risk (AVaR). The third one has been used relatively recently in the financial practice – the expectile risk measure (ERM). Assume that the log-returns of the considered asset – stock, index, portfolio etc. – follow some distribution. We shall preserve for convenience the notations $f(\cdot)$, $F(\cdot)$, $\bar{F}(\cdot)$, $q(\cdot)$, and $\Psi(\cdot)$ for the probability density function (PDF), cumulative distribution function (CDF), the complementary cumulative distribution function (CCDF), the quantile function, and the characteristic function of the distribution. If we need a further clarification to which random variable these functions are associated, say ξ , we shall use a lower sub-script – $f_\xi(\cdot)$, $F_\xi(\cdot)$, $\bar{F}_\xi(\cdot)$, $q_\xi(\cdot)$, and $\Psi_\xi(\cdot)$. Also, we shall parametrize the risk measures as $VaR(\epsilon)$, $ES(\epsilon)$, and $ERM(\epsilon)$ or $VaR(\epsilon; \xi)$, $ES(\epsilon; \xi)$, and $ERM(\epsilon; \xi)$ if needed. We shall establish the relations between all these measures as well as their relations to the characteristic function. The VaR measure is just the opposite of the quantile function

$$VaR(\epsilon; \xi) = -q_\xi(\epsilon) = -F^{-1}(\epsilon), \quad (2.1)$$

where the CDF – $F(\epsilon)$ – can be derived through Proposition 3.1 below. The expected shortfall is defined as averaged VaR , namely

$$ES(\epsilon; \xi) = \frac{\int_0^\epsilon VaR(u; \xi) du}{\epsilon}. \quad (2.2)$$

We shall use the notations x^+ and x^- for $\max(x, 0)$ and $\max(-x, 0)$. We describe below the ES through the truncated expectations.

Proposition 2.1. *The following statement holds*

$$ES(\epsilon; \xi) = \frac{E[(\xi - q(\epsilon))^-]}{\epsilon} - q(\epsilon). \quad (2.3)$$

Proof. Using the change of variables $s = q(t)$ or equivalently $t = F(s)$, we derive

$$\begin{aligned} ES(\epsilon; \xi) &= -\frac{1}{\epsilon} \int_0^\epsilon q(t) dt = -\frac{1}{\epsilon} \int_{-\infty}^{q(\epsilon)} s dF(s) = -\frac{E[\xi I_{\xi < q(\epsilon)}]}{\epsilon} \\ &= \frac{E[(\xi - q(\epsilon))^- - q(\epsilon) I_{\xi < q(\epsilon)}]}{\epsilon} \\ &= \frac{E[(\xi - q(\epsilon))^-]}{\epsilon} - q(\epsilon), \end{aligned} \quad (2.4)$$

which finishes the proof. $\square$

The ϵ -expectile term is defined firstly in Newey and Powell (1987) as the solution of the following square optimization problem

$$\arg \min_{x \in \mathbb{R}} \left\{ E[(\xi - x)^+]^2 + (1 - \epsilon)(\xi - x)^- \right\}. \quad (2.5)$$

Note that it is well defined only for random variables with finite second moments. We shall define the $ERM(\epsilon; \xi)$ as the opposite of expectile (2.5). For the bivariate and multivariate expectile see Cascos and Ochoa (2021). The analogies with the quantile function can be viewed if we present the later as the solution of the linear optimization problem

$$q_\xi(\alpha) = \arg \min_{x \in \mathbb{R}} \left\{ E[\alpha(\xi - x)^+ + (1 - \alpha)(\xi - x)^-] \right\}. \quad (2.6)$$

Simple calculus lead to the following equation presentation of the ERM:

Proposition 2.2. *ERM($\epsilon; \xi$) is the unique solution of the equation w.r.t. the variable x*

$$\epsilon E[(\xi + x)^+] = (1 - \epsilon) E[(\xi + x)^-]. \quad (2.7)$$

Proof. The result can be derived differentiating formula (2.5). $\square$

Remark 2.1. The expectile itself is the solution of the equation

$$\epsilon E[(\xi - x)^+] = (1 - \epsilon) E[(\xi - x)^-]. \quad (2.8)$$

Let us discuss the way we extract the three risk measures from a historical sample. Let us have n observations of the log-returns r_i , $i = 1, 2, \dots, n$. We shall denote by $r_{\min}$ and $r_{\max}$ the smallest and largest of them, respectively. The quantile function, and thus the measure VaR , can be obtained by the level which contains the 100 ϵ % of observations in the left tail. Usually, ES is obtained by averaging VaR . We suggest the following procedure for deriving the ERM through formula (2.7) – note that this method can be used for deriving the ES too via Proposition 2.1. The following lemma holds for the truncated expectations.

Lemma 2.1. *The truncated at the interval (a, b) expectation, if it exists, can be derived as*

$$E[(\xi - x) I_{a < \xi < b}] = (b - x) F(b) - (a - x) F(a) - \int_a^b F(y) dy. \quad (2.9)$$

Proof. The desired result follows after an integration by parts. $\square$

We estimate numerically the risk measures from the empirical data in the following way.

1. We calculate the empirical CDF at the point x as the number of observations lower than x divided by the total observations number

$$F_{\text{empirical}}(x) = \frac{\#\{r_i < x\}}{n}. \quad (2.10)$$

Note that

$$\begin{aligned} F_{\text{empirical}}(r_{\min}) &= 0 \\ F_{\text{empirical}}(r_{\max}) &= 1. \end{aligned} \quad (2.11)$$

2. Having in mind that the interval for the log-returns is $(r_{\min}, r_{\max})$ and using Lemma 2.1 and Eqs. (2.11) we derive for the truncated expectations

$$\begin{aligned} E[(\xi - x)^+] &= (r_{\max} - x) - \int_x^{r_{\max}} F_{\text{empirical}}(y) dy \\ E[(\xi - x)^-] &= \int_{r_{\min}}^x F_{\text{empirical}}(y) dy. \end{aligned} \quad (2.12)$$

3. We obtain the quantile function, and as a consequence the VaR , as $q(x) = \inf\{y, F_{\text{empirical}}(y) < x\}$. The ES measure can be derived using Proposition 2.1 and the second part of Eqs. (2.12). The expectile and thus the ERM can be derived through Eqs. (2.7) and (2.8) having in mind presentations (2.12).

3. Deriving the theoretical risk measures

Suppose that the log-asset price is presented by a stochastic processes for which we know only the characteristic function, but not the PDF. We need the cumulative distribution function to derive the measure VaR . On the other hand, we need the truncated expectations to obtain the ES and ERM through Propositions 2.1 and 2.2. Below we find these terms via the characteristic function.

3.1. Necessary terms

We shall use the following form of the Fourier transform which agrees with the definition of the characteristic function

$$\hat{f}(u) \equiv (Ff)(x) = \int_{-\infty}^{\infty} e^{ixu} f(x) dx. \quad (3.1)$$

As usually, we shall notate the real and imaginary part of a complex number by the symbols $\Re(\cdot)$ and $\Im(\cdot)$, respectively. Let us formulate the following version of the Fourier inverse formula.

Theorem 3.1. Suppose that for some random variable ξ the constants $a_1 < a_2$ satisfy the condition

$$E[e^{-a_{1,2}\xi}] < \infty. \quad (3.2)$$

Then for every $b \in (a_1, a_2)$, the PDF of ξ can be obtained via its characteristic function as

$$f_\xi(x) = (F^{-1}\Psi_\xi)(x) = \frac{1}{2\pi} \int_{ib-\infty}^{ib+\infty} e^{-ixu} \Psi_\xi(u) du. \quad (3.3)$$

Proof. We refer to Theorem 7.1.1. of Lukacs (1970). $\square$

Remark 3.1. Note that the characteristic function $\Psi_\xi(z)$ is analytical in the whole strip $a_1 < \Im(z) < a_2$ and hence we may use the Cauchy's contour theorem.

Remark 3.2. We can think that $a_1 \leq 0 \leq a_2$, since condition (3.2) holds for $a = 0$. Note that Theorem 3.1 can be rewritten for arbitrary function $f(\cdot)$, not only for PDF's. In this case the inequality $a_1 \leq 0 \leq a_2$ is not obligatory.

The following presentations for the CDF via the characteristic function hold.

Proposition 3.1. Let the constants $a_1 \leq 0 \leq a_2$ satisfy condition (3.2) and $b \in (a_1, a_2)$. The following three cases arise w.r.t. the sign of b .

1. If $b > 0$, then

$$F_\xi(x) = \frac{e^{bx}}{\pi} \int_0^{+\infty} \Im \left(\frac{e^{-ixv} \Psi_\xi(v+ib)}{-v-ib} \right) dv. \quad (3.4)$$

2. If $b < 0$, then

$$F_\xi(x) = 1 - \frac{e^{bx}}{\pi} \int_0^{+\infty} \Im \left(\frac{e^{-ixv} \Psi_\xi(v+ib)}{v+ib} \right) dv. \quad (3.5)$$

3. We have in the limiting case $b = 0$

$$F_\xi(x) = \frac{1}{2} + \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_\epsilon^{+\infty} \Im \left(\frac{e^{-ixv} \Psi_\xi(v)}{v} \right) dv. \quad (3.6)$$

Proof. The proof is presented in Appendix A.1. $\square$

The next terms we need are the truncated expectations. The following proposition describes them.

Proposition 3.2. If the constants $a_1 \leq 0 \leq a_2$ satisfy condition (3.2) and $a_1 \leq b_1 < 0 < b_2 \leq a_2$, then

$$\begin{aligned} E[(\xi - x)^-] &= -\frac{1}{2\pi} \int_{ib_2-\infty}^{ib_2+\infty} \frac{\Psi_\xi(u) e^{-ixu}}{u^2} du \\ &= -\frac{e^{b_2x}}{\pi} \int_0^{+\infty} \Re \left(\frac{e^{-ixv} \Psi_\xi(v+ib_2)}{(b_2-iv)^2} \right) dv \\ E[(\xi - x)^+] &= -\frac{1}{2\pi} \int_{ib_1-\infty}^{ib_1+\infty} \frac{\Psi_\xi(u) e^{-ixu}}{u^2} du \\ &= -\frac{e^{b_1x}}{\pi} \int_0^{+\infty} \Re \left(\frac{e^{-ixv} \Psi_\xi(v+ib_1)}{(b_1-iv)^2} \right) dv. \end{aligned} \quad (3.7)$$

Something more, the following presentations also hold.

$$\begin{aligned} E[(\xi - x)^-] &= \frac{i\Psi'_\xi(0) + x}{2} - \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_\epsilon^{+\infty} \Re \left(\frac{e^{-ixv} \Psi_\xi(v)}{v^2} \right) dv. \\ E[(\xi - x)^+] &= -\frac{i\Psi'_\xi(0) + x}{2} - \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_\epsilon^{+\infty} \Re \left(\frac{e^{-ixv} \Psi_\xi(v)}{v^2} \right) dv. \end{aligned} \quad (3.8)$$

Proof. The proof is provided in Appendix A.2. $\square$

3.2. Theoretical risk measures

Based on Proposition 3.1 for the CDF we can derive the VaR through converting and taking the opposite. On the other hand, using Proposition 3.2 for the truncated expectations we shall prove the following theorem for the ES and ERM.

Theorem 3.2. Let $0 < b_2 < a_2$ and $\epsilon < \frac{1}{2}$. Then the expected shortfall can be derived as

$$\begin{aligned} ES(\epsilon; \xi) &= -q_\xi(\epsilon) - \frac{e^{b_2q_\xi(\epsilon)}}{\epsilon\pi} \int_0^{+\infty} \Re \left(\frac{e^{-iq_\xi(v)v} \Psi_\xi(v+ib_2)}{(b_2-iv)^2} \right) dv \\ &= \frac{(1-2\epsilon)q_\xi(\epsilon) + i\Psi'_\xi(0)}{2\epsilon} - \frac{1}{\epsilon\pi} \lim_{\delta \rightarrow 0} \int_\delta^{+\infty} \Re \left(\frac{e^{-iq_\xi(v)v} \Psi_\xi(v)}{v^2} \right) dv. \end{aligned} \quad (3.9)$$

The ϵ -expectile can be obtained as the solution of the following equation w.r.t. the variable x

$$x = \frac{2(1-2\epsilon)}{\pi} \lim_{\delta \rightarrow 0} \int_\delta^{+\infty} \Re \left(\frac{e^{-ixv} \Psi_\xi(v)}{v^2} \right) dv. \quad (3.10)$$

Alternatively, if $a_1 < b_1 < 0 < b_2 < a_2$, then we obtain an equivalent to (3.10) equation:

$$\int_0^{+\infty} \Re \left(\frac{(1-\epsilon)e^{(b_2-b_1)x} (b_1-iv)^2 \Psi_\xi(v+ib_2) - \epsilon(b_2-iv)^2 \Psi_\xi(v+ib_1)}{(b_1-iv)^2 (b_2-iv)^2} \right) e^{-ixv} dv = 0. \quad (3.11)$$

Consequently, the expectile based risk measure, $ERM(\epsilon; \xi)$, is the opposite of the solution of Eqs. (3.10) and (3.11).

Proof. Formula (3.9) for the expected shortfall follows from Propositions 2.1 and 3.2. Eqs. (3.10) and (3.11) for the expectile can be obtained through Eqs. (2.7), (3.7) and (3.8), respectively. $\square$

3.3. Stating the models

Suppose that the financial asset we consider is modeled under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ which satisfies the usual conditions – the filtration is right continuous and complete. Also, let three stochastic processes be defined on this space – two correlated Brownian motions B_t and $\tilde{B}_t$, and a Lévy process Y_t . We shall compare the risk behavior which generate five different models – (1) Black–Scholes model, (2) exponential tempered stable model, (3) Heston model (stochastic volatility model without jumps) (4) Bates model (stochastic volatility model with compound Poisson normally distributed jumps), and (5) stochastic volatility model with tempered stable jumps. Thus we have the following parametrization of the models:

3.3.1. Black–Scholes model

The asset is presented by the log-normal process which dynamics is

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \quad (3.12)$$

As obvious μ and σ are the asset return and risk, respectively.

3.3.2. Exponential tempered stable model

Now the asset is presented by $S_t = S_0 e^{Y_t}$, where Y_t is a tempered stable process. This is a seven parameters distribution – drift and three parameters for each of the negative and positive jumps. Thus its Lévy measure is

$$\nu(u) = \frac{c_1}{(-u)^{1+\alpha_1}} e^{\lambda_1 u} I_{u < 0} + \frac{c_2}{u^{1+\alpha_2}} e^{-\lambda_2 u} I_{u > 0} \quad (3.13)$$

for some positive constants $c_{1,2}$, $\lambda_{1,2}$, and $\alpha_{1,2} \in (0, 2) - c_{1,2}$, λ_1 , and α_1 describe the negative jumps and c_2 , λ_2 , and α_2 are related to the positive ones. Note that Lévy measure (3.13) is an exponentially truncated Lévy measure of a stable distribution – this way the tempered stable one exhibits finite second moments. Note also that the constants $a_1 = -\lambda_2$ and $a_2 = \lambda_1$ satisfy the conditions of Section 3.1 and thus all proven propositions hold.

The characteristic exponent of this distribution can be written as

$$\psi_{TS}(u) = i\mu u + \psi_1(u) + \psi_2(u), \quad (3.14)$$

where

$$\psi_1(u) = \begin{cases} \Gamma(-\alpha_1) \lambda_1^{\alpha_1} c_1 \left(\left(1 + \frac{iu}{\lambda_1}\right)^{\alpha_1} - 1 - \frac{iu\alpha_1}{\lambda_1} \right), & \text{if } \alpha_1 \neq 1 \\ -iuc_1 + c_1 (\lambda_1 + iu) \ln \left(1 + \frac{iu}{\lambda_1}\right), & \text{if } \alpha_1 = 1 \end{cases}$$

$$\psi_2(u) = \begin{cases} \Gamma(-\alpha_2) \lambda_2^{\alpha_2} c_2 \left(\left(1 - \frac{iu}{\lambda_2}\right)^{\alpha_2} - 1 + \frac{iu\alpha_2}{\lambda_2} \right), & \text{if } \alpha_2 \neq 1 \\ iuc_2 + c_2 (\lambda_2 - iu) \ln \left(1 - \frac{iu}{\lambda_2}\right), & \text{if } \alpha_2 = 1. \end{cases} \quad (3.15)$$

This way the characteristic function is $\Psi_{TS}(u) = e^{\psi_{TS}(u)}$.

3.3.3. Heston model

The asset is driven by the two-dimensional stochastic differential equation

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dB_t \\ dV_t &= \xi(\eta - V_t) dt + \theta \sqrt{V_t} d\tilde{B}_t. \end{aligned} \quad (3.16)$$

The Brownian motions B_t and $\tilde{B}_t$ are correlated with the coefficient ρ . Here the variable ξ is the speed of reversion to the value η ; θ is the volatility of the volatility. Thus the Heston model is driven by five parameters μ , ξ , η , θ , and ρ . The log-price can be written as

$$d(\log(S_t)) = \left(\mu - \frac{V_t}{2}\right) dt + \sqrt{V_t} dB_t. \quad (3.17)$$

Hence, the characteristic function of the log-returns can be presented as

$$\Psi_{Heston}(t, x, v; u) = e^{C(t,u) + D(t,u)v}, \quad (3.18)$$

where

$$\begin{aligned} C(s, u) &= \mu u s + \frac{\xi \eta}{\theta^2} \left[(\xi - \rho \theta u i + d) s - 2 \log \left(\frac{1 - g \exp(d_j s)}{1 - g} \right) \right] \\ D(s, u) &= \frac{\xi - \rho \theta u i + d}{\theta^2} \frac{1 - \exp(ds)}{1 - g \exp(ds)} \\ g &= \frac{\xi - \rho \theta u i + d}{\xi - \rho \theta u i - d} \\ d &= \sqrt{(\rho \theta u i - \xi)^2 + \theta^2 (u i + u^2)}. \end{aligned} \quad (3.19)$$

For more details see Heston (1993).

3.3.4. Bates model

The stochastic volatility model of Bates is defined as the Heston model with an added jump correction, presented by a compound Poisson process with log-normally distributed jumps. If we denote it by $\bar{X}_t$, then the asset dynamics is

$$\begin{aligned} dS_t &= \mu^S S_t dt + \sqrt{V_t} S_t dB_t + S_t d\bar{X}_t \\ dV_t &= \xi(\eta - V_t) dt + \theta \sqrt{V_t} d\tilde{B}_t. \end{aligned} \quad (3.20)$$

The support of $\bar{X}_t$ is the interval $(-1, +\infty)$. This guarantees the positiveness of the asset price process S_t . The intensity is denoted by λ and the jump sizes are parametrized as $N\left(\log(1+k) - \frac{\delta^2}{2}, \delta^2\right)$. Hence, if the jump distribution is denoted by Z , then

$$\log(1+Z) \sim N\left(\log(1+k) - \frac{\delta^2}{2}, \delta^2\right). \quad (3.21)$$

Thus the Bates model depends on eight parameters μ , ξ , η , θ , ρ , λ , k , and δ . The log-price can be written as

$$d(\log(S_t)) = \left(\mu - \frac{V_t}{2}\right) dt + \sqrt{V_t} dB_t + dX_t. \quad (3.22)$$

The process X_t is compound Poisson with intensity λ and (k, δ) -normally distributed jumps. The characteristic function of the log-price can be presented as a product of two parts – Heston term (3.18) multiplied by the jump characteristic function

$$\Psi(t, x, v; u) = \Psi_{Heston}(t, x, v; u) e^{t\psi_J(u)}, \quad (3.23)$$

where in this case $\psi_J(u)$

$$\psi_J(u) = \lambda \left(e^{-\frac{\delta^2 u^2}{2} + i\left(k - \frac{\delta^2}{2}\right)u} - 1 \right). \quad (3.24)$$

3.3.5. Stochastic volatility/tempered stable model

This model is proposed by Zaeviski et al. (2014) and it differs from the Bates model by the style of the added jumps – now they are presented by an infinite activity process. Thus the formulas for the asset price and the log-returns, (3.20) and (3.22), still hold but X_t is a tempered stable process.⁵ Its parameters are $c_{1,2}$, $\lambda_{1,2}$, and $\alpha_{1,2}$. This way the characteristic function of the asset is presented by formula (3.18), but now the characteristic exponent of the jump process is

$$\psi_J(u) = \psi_1(u) + \psi_2(u), \quad (3.25)$$

where $\psi_1(\cdot)$ and $\psi_2(\cdot)$ are given in formulas (3.15). This model is driven by eleven parameters μ , ξ , η , θ , ρ , $c_{1,2}$, $\lambda_{1,2}$, and $\alpha_{1,2}$.

4. Calibration methodology and numerical results

We present in this section a numerical comparison between the mentioned above five theoretical models. We first calibrate these models to the empirical data and then we analyze the risk measures VaR , ES , and ERM they generate w.r.t. the empirical ones.

4.1. Historical data

We use historical data for the S&P500 index for the period between October 6, 2004 and July 01, 2022 – totally 4466 observations. We use the daily close values. We move the historical sample with length 1000 days from September 25, 2008 to July 01, 2022. That way the period between October 6, 2004 and September 24, 2008 is the first sample. These dates are chosen this way because the theoretical models are calibrated to the date September 25, 2008 in Zaeviski et al. (2014). Thus we can use the derived parameters' values for the initial point for the calibrating minimization for the first sample – it is well known that the initial point is very important when we minimize a multivariate function. We use the derived parameters for the initial point when calibrate the models to the next day sample. That way we have totally 3466 samples.

4.2. Calibrating methodology

Let the S&P500 index values for a particular sample are $S_1, S_2, \dots, S_N$, $i = 1, 2, \dots, N-1$. We derive the log-returns in the usual manner $r_i = \ln(S_{i+1}) - \ln(S_i)$. The empirical probability density function is derived by the following standard algorithm. We divide the interval $(-0.25, 0.25)$ into $M = 250$ bins with length $\Delta = 0.002$. The interval is choose this way because the higher deviation in the S&P500 index log-returns is -0.2290 at the Black Monday, October 19, 1987. Let n_i , $i = 1, 2, \dots, 250$, be the number of log-returns belonging to each bin. The empirical PDF in the center of the i th bin is approximated by $\frac{n_i}{N\Delta}$.

⁵ The form of $\bar{X}_t$ can be viewed in appendix A of Zaeviski et al. (2014).

To calibrate the models we also need the theoretical PDF for the different models. Except for the Black–Scholes model, the PDFs are not available, hence we must derive them through the characteristic functions. We invert them using the fast Fourier transform (FFT) algorithm. To find a relatively good approximation we derive the PDFs at 500 points uniformly taken at the interval $(-0.5, 0.5)$ – the length of the bins is again $\Delta = 0.002$. Finally, we derive the PDF using the cubic spline approach.

One more problem stands when we calibrate the three stochastic volatility models. They depend on an unobservable parameter, namely the initial volatility. To overcome this problem we apply a method firstly suggested by Drăgulescu and Yakovenko (2002) and later used in Zaeviski et al. (2014). We average the characteristic function via the stationary distribution of the volatility driven by stochastic differential Eq. (3.16), the second one. Since the volatility is expressed by a CIR process, see Cox et al. (1985), the stationary distribution is a gamma one with parameters $c = \frac{2\xi\eta}{\theta^2}$ and $\alpha = \frac{2\xi}{\theta^2}$. Its PDF is

$$f_\gamma(u) = \frac{\alpha^c}{\Gamma(c)} u^{c-1} \exp(-\alpha u) I_{u \geq 0}. \quad (4.1)$$

Thus the averaged characteristic function is

$$\begin{aligned} \Psi_{Heston}^{av}(t, x, v; u) &= \int_0^\infty \Psi_{Heston}(t, x, v; u) f_\gamma(v) dv \\ &= \int_0^\infty e^{C(t,u)+D(t,u)v+t\psi(u)} f_\gamma(v) dv \\ &= e^{C(t,u)+t\psi(u)} \psi_\gamma(-iD(t,u)), \end{aligned} \quad (4.2)$$

where $\psi_\gamma(\cdot)$ is the characteristic function of the gamma distribution

$$\psi_\gamma(u) = \left(1 - \frac{iu}{\alpha}\right)^{-c} \equiv \left(1 - \frac{i\theta^2 u}{2\xi}\right)^{-\frac{2\xi\eta}{\theta^2}}. \quad (4.3)$$

Note that for all s and u , $\Re D(s, u)$ is negative, and therefore

$$\int_0^\infty \exp(\Re D(t, u) v) f_\gamma(v) dv < \infty. \quad (4.4)$$

The method we suggest for calibration is based on the least square errors (LSqE) approach. Let us denote by Y the set of all possible values of the parameters of some theoretical model and by γ some particular realization, $\gamma \in Y$. We modify the usual least square error (LSqE) criterion for minimization to

$$\min_{\gamma \in Y} \left\{ \sum_{i=1}^{250} (\ln(p_{th,i} + 1) - \ln(p_{emp,i} + 1))^2 \right\}, \quad (4.5)$$

where $p_{th,i}$ and $p_{emp,i}$ are the corresponding values of the theoretical PDF based on the parameter's set γ , and the empirical PDF. We use the logarithm to give relatively equal impact of the terms in the distribution's center and the tails. The unit is added to the PDFs because some of the empirical values may be equal to zero.

4.3. Validation and comparison of the fits

Once we calibrate the models to the moving empirical data with length of 1000 we can prepare many experiments to validate the goodness of fit. The approach we suggest has some similarities with the maximum likelihood method but applied to forecasting. Let us denote by $p(x; i)$ the calibrated PDF for some model at the i th date. We shall use the following forecast function

$$\prod_{i=1}^{N-j} p(r_{i+j}, i) \quad (4.6)$$

or the related logarithmic one

$$\sum_{i=1}^{N-j} \ln(p(r_{i+j}, i)). \quad (4.7)$$

This way we may evaluate the forecast of the fit at the date i for the value at the date $i+j$. We calculate the values which function (4.7)

generates for $j = 1$ (one day), 5 (one week), and 20 (approximately one month). The results are presented in Table 3. We provide in this table also the averaged non-logarithmic values calculated by the formula

$$\frac{\sum_{i=1}^{N-j} p(r_{i+j}, i)}{N-j}. \quad (4.8)$$

We can see that all models (except the Black–Scholes one) produce relatively equal forecasts. Note that a larger value means a better fit. Looking at the averaged values we see that the tempered stable models lead to a better forecast regardless of the presence or not of the stochastic volatility. The forecasts different models produce are presented at Fig. 2 too – sub- Figs. 2(a)–2(e). On the axes are the support of the log-returns and the time presented by years. The surface presents the calibrated PDF's w.r.t. all days between September 25, 2008 to July 01, 2022, whereas the red points are the actually happened log-returns with one day lag.

The calibrated parameters for the different models based on the whole used data – from October 6, 2004 to July 01, 2022 – are presented in Table 2. The corresponding PDFs they lead to can be seen at Fig. 2(f). The conclusion that all models (except the Black–Scholes one) describe relatively similarly the empirical data is confirmed again. We can observe that the correlation coefficients for the stochastic volatility models are negative – -0.2021 , -0.4189 , and -0.6000 for the Heston, Bates and SV/TS models, respectively. This confirms some market phenomena as the leverage effect. We can see also that the normally distributed jumps in the Bates model have a relatively large intensity, 9.5952, but a low jump size and deviation – 0.0022 and 0.0332. This leads to a behavior which is characteristic rather of an infinite activity Lévy process than the compound Poisson one. Note also that the criterion we choose for our calibration methodology given in Eq. (4.5) leads to relatively large deviations in the distribution center but the fit is very strong in the tails. The errors that occurred are presented at the second line of Table 3. All models except the Black–Scholes one (error 7.4516) give errors near one. We can see again that the tempered stable process produces more realistic behavior. The error for the exponential tempered stable model is 1.1083 and for the stochastic volatility model with a tempered stable correction is 1.0809. For a comparison – the Heston and Bates deviations are 1.1845 and 1.1722, respectively.

The daily deviations which the models produced are presented at Fig. 3(d). Together we plot the values of the corresponding log-returns multiplied by 50 for a good visualization. Excluding the Black–Scholes model which produces higher deviations, the rest models have similar errors. The averaged errors are presented in Table 3, the first line. We can see that the stochastic volatility model with tempered stable jumps gives the smallest deviation: 4.2394. The other jump models, namely the Bates one and the exponential tempered stable model, have averaged errors of 4.3096 and 4.3051, respectively. The Heston model which exhibits continuous trajectories produces an averaged deviation with value of 4.5368. We can conclude that the jump models describe more realistically the empirical observations. In addition, the Black–Scholes' error is 9.7756 – hence we have validated the well known fact that the Gaussian assumption for the log-returns is not supported by the historical data.

Another conclusion we can make from Fig. 3(d) is that the larger deviations appear in the periods of high volatility. This observation can be seen in the behavior of all models.

4.4. Risk measures – VaR, ES, and ERM

The question for deriving the empirical measures is considered in Section 2. Theoretical values are derived through the already calibrated PDFs. For the VaR we use the CDF obtained by the cubic spline method applied to the cumulative sum of the PDF and then inverted. The ES is obtained by averaging (integrating) the VaR function. We find the ERM using Proposition 2.2, see also Remark 2.1.

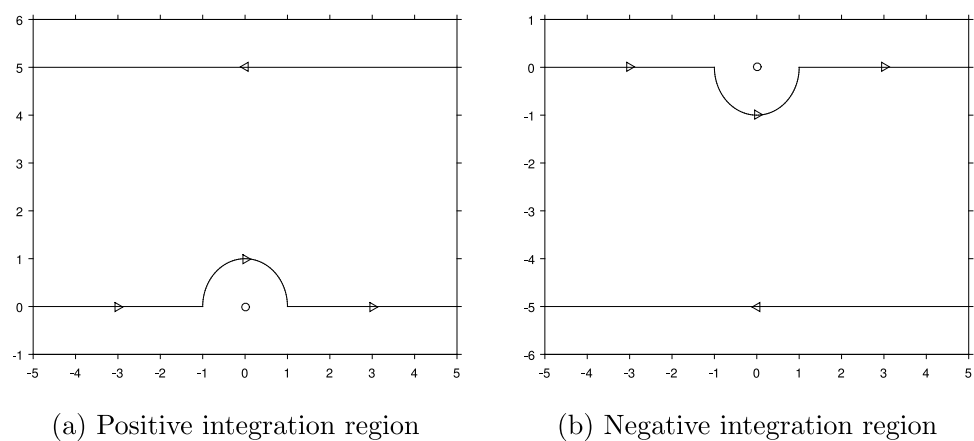


Fig. 1. Integration regions.

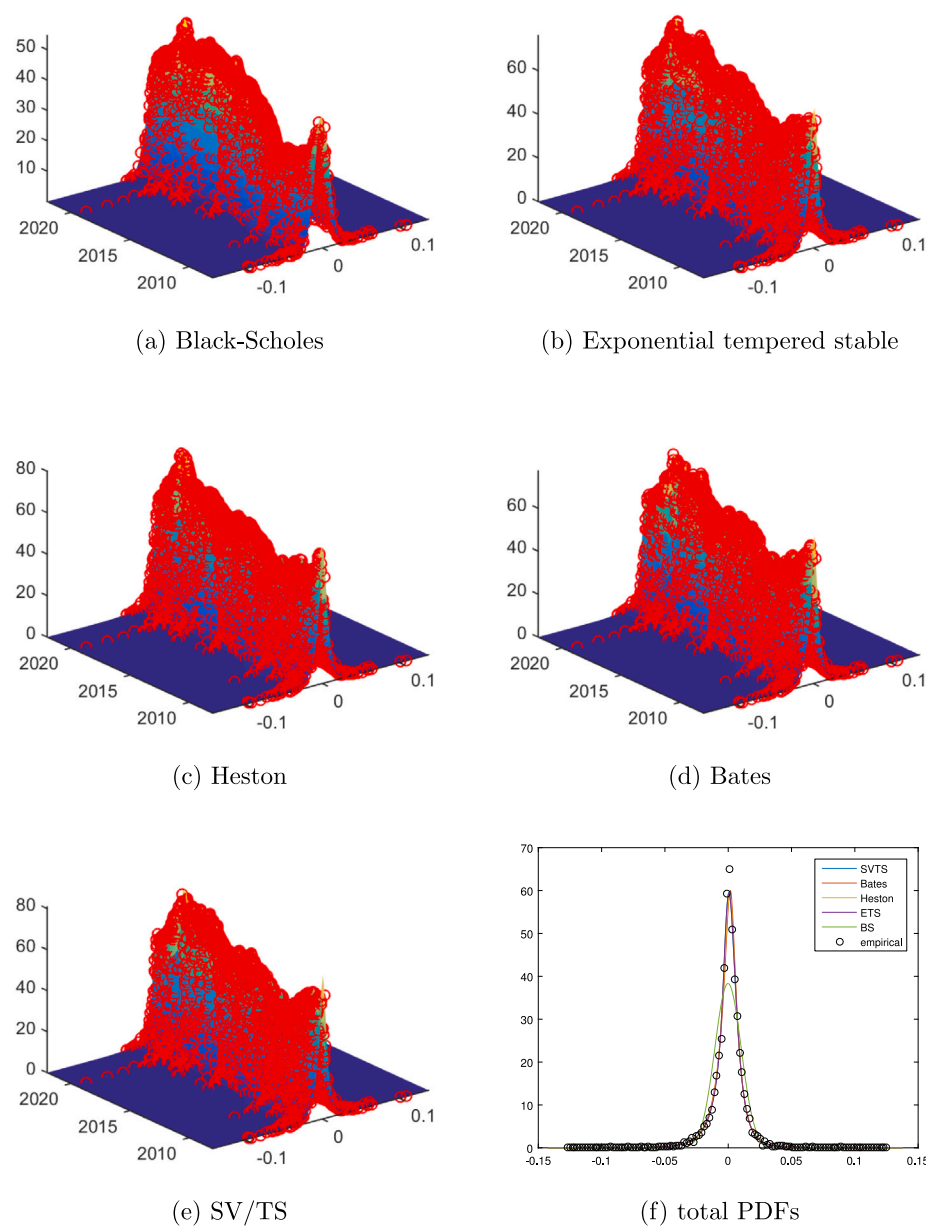


Fig. 2. Forecasts and the calibrated PDF's.

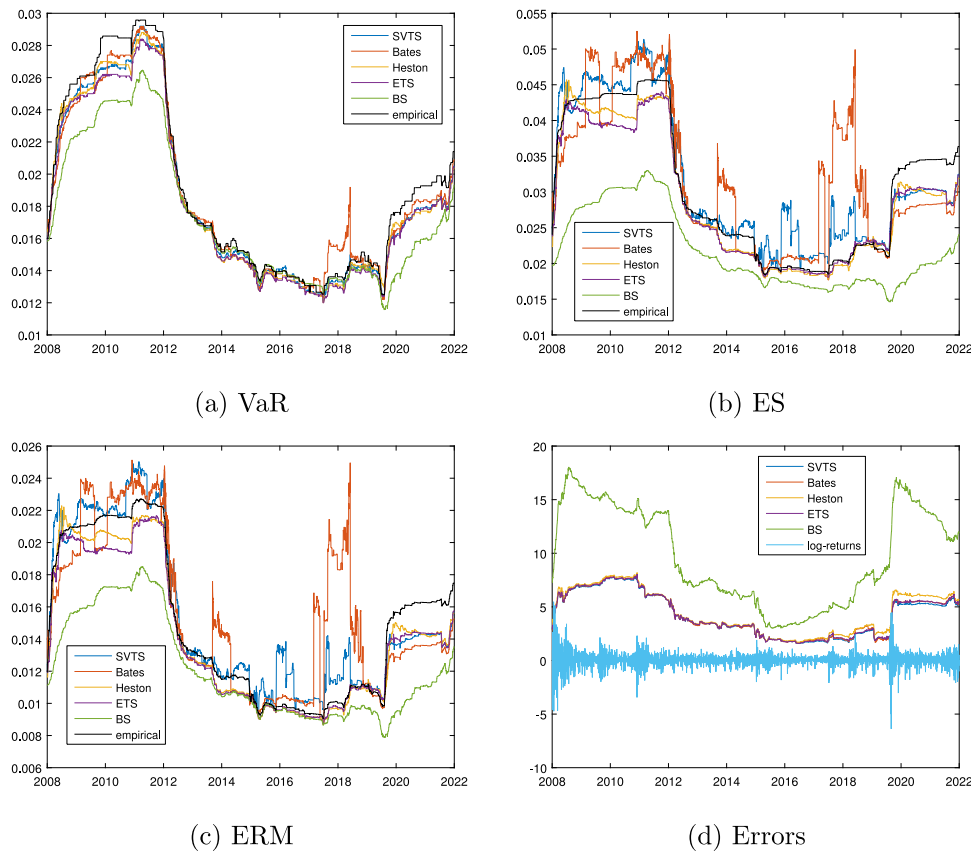


Fig. 3. Risk measures, estimation errors, and log-returns.

Table 1

The total empirical VaR, ES, and ERM and the corresponding ratios.

Parameter	VaR	ES	ERM	$k_1 = VaR/ES$	$k_2 = VaR/ERM$
$\epsilon = 0.01$	0.0366	0.0556	0.0290	0.6577	1.2634
$\epsilon = 0.02$	0.0295	0.0437	0.0225	0.6753	1.3148
$\epsilon = 0.03$	0.0244	0.0381	0.0190	0.6414	1.2837
$\epsilon = 0.04$	0.0210	0.0343	0.0167	0.6129	1.2548
$\epsilon = 0.05$	0.0186	0.0313	0.0150	0.5927	1.2335

As we mentioned above, it is very important for the financial institutions different risk measures to return relatively closed results. The currently adopted method for this adjustment is based on changing the levels at which the measures are considered. The main disadvantage of this approach is that we lose (or add) a tail information changing the level. Alternatively, we suggest the use of correction coefficients between the VaR on the one hand and the ES and the ERM on the other. The coefficients we introduce are obtained as the ratios between the empirical VaR and ES and ERM calculated for the whole data. We shall denote them by $k_1 = VaR/ES$ and $k_2 = VaR/ERM$. The derived values of the risk measures together with the corresponding ratios are presented in Table 1. The used levels are $\epsilon \in \{0.01, 0.02, 0.03, 0.04, 0.05\}$. We can see that the values of k_1 and k_2 are relatively stable w.r.t. the level ϵ and hence we assume that $k_1 = 0.6$ and $k_2 = 1.25$. Alternatively, the coefficients k_1 and k_2 can be considered as functions of ϵ . The exact values can be adjusted via additional tests based on many other indexes, assets, etc. if the suggested approach would be accepted in the financial practice.

The original values of the risk measures for the five models – stochastic volatility/tempered stable jumps (SV/TS), Bates, Heston, exponential tempered stable (ETS), and Black–Scholes (BS) models – are compared at Fig. 3. We can see that all models follow the empirical VaR -measure relatively close. Let us note that the Bates model has a

large deviation near 2018 year. A possible explanation is that this is due to a shortcoming of a calibration methodology. The same observation exists for the other models too but to a lesser extent. Nonetheless, it can be seen that the approach we suggest captures the empirical values after some time. On the other hand, we can restart the algorithm at the current moment if it returns a large error since we re-calibrate daily. This is due to the large flexibility of our method and its weak time coherence. In such a way the new parameters may describe more efficiently the eventual new market realities. The same conclusion but strengthened can be made for the other two risk measures – the expected shortfall and the expected return measure. This can be explained by the facts that (1) the ES is integrated VaR -s and (2) the ERM is based on the more information than VaR . We have to mention that the Black–Scholes model follows the trend of all risk measures, but it leads to some underestimation. On the other hand, this model has a great advantage – it depends only on two parameters and thus its calibration is very easy. The averaged deviations for all models are reported in Table 3 – the third, fourth, and fifth lines. We can confirm the observation made for the fits that the existence of tempered stable jumps leads to fine results in the presence or not of stochastic volatility. Also, the Heston and the Bates model lead to good results too mentioning the large deviation in the ES risk measure for the Bates model – the explanation for it is provided above. Nonetheless, the stochastic volatility model with tempered stable jump correction produces best results. The price which we have to pay for this is the large number of the parameters and some calibration difficulties it leads to.

We present the corrected by the coefficients $k_1 = 0.6$ and $k_2 = 1.25$ risk measures at Fig. 4. It can be seen that excluding the mentioned above some rare discrepancies, which can be avoided by restarting the calibration algorithm when they happen, all models lead to a similar and parallel behavior of the all risk measures. This observation is true

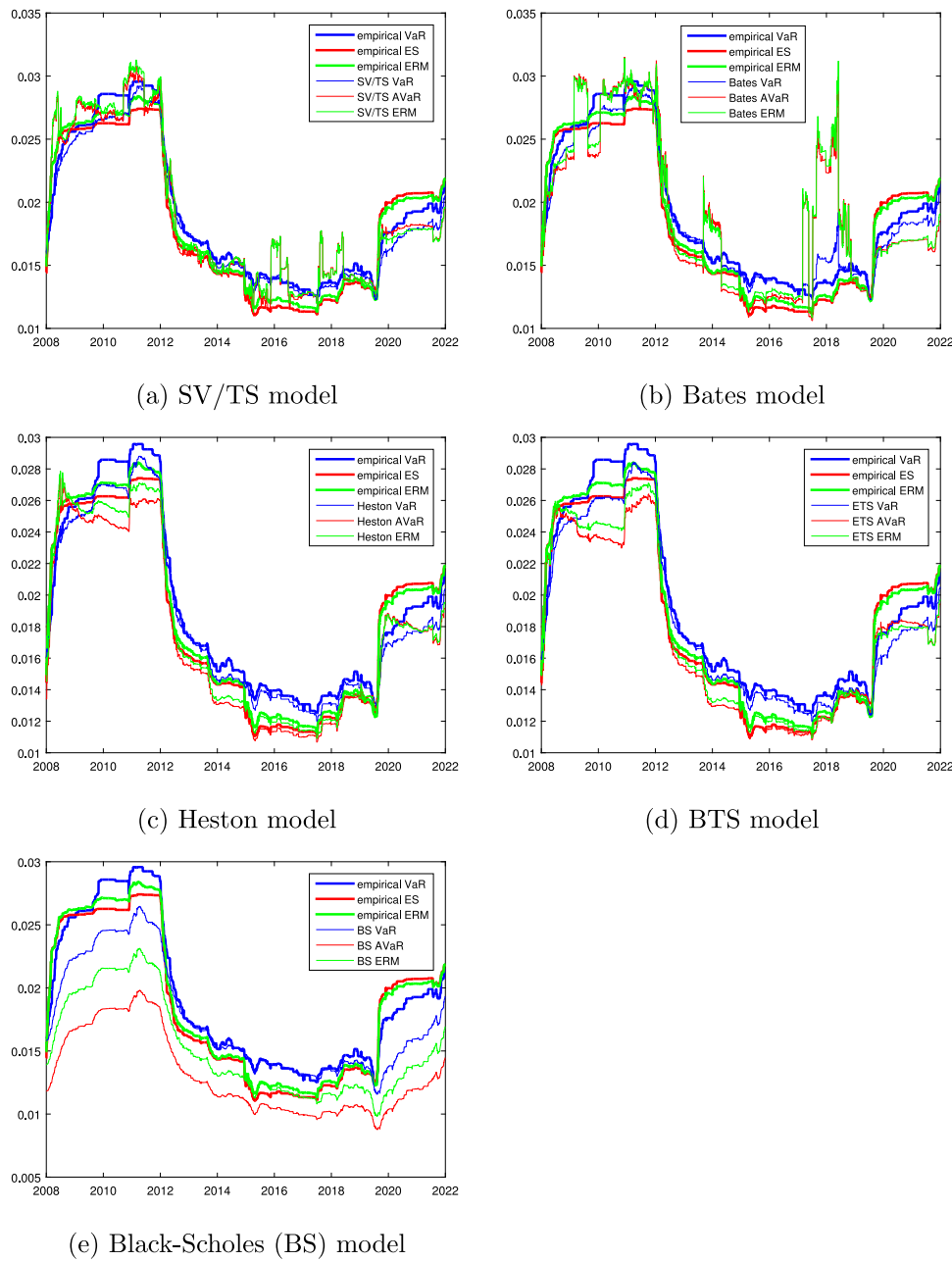


Fig. 4. Modified risk measures.

for the Black–Scholes model too with the remark of its underestimation. All these findings strongly support our suggestion to introduce the correction coefficients between the VaR -measure on the one hand and ES and ERM on other.

Finally, we calculate the risk measures which the already calibrated models for the period between October 6, 2004 and July 01, 2022 lead to. Remind that the parameters values are reported in Table 2. The results for the levels 0.05, 0.025, and 0.01 are presented in Table 4. The left values for the expected shortfall and the expectile risk measure are the original ones, whereas the right values are corrected by the coefficients k_1 and k_2 . We can see that the results for the VaR are very closed to real ones for all models except for the Black–Scholes one, which again produces significantly lower but predictable results. For the rest two measures the results are again admissible and confirm the previous conclusions.

What we see on both Figs. 3 and 4 is a common pattern for all risk measures: three regimes are well discerned. From the begin of

the observation period to late-2012 we witness the aftermath of the global financial crisis. Since 2020 a new plateau is formed to reflect the COVID-19 impact on the financial markets, although at a lower level than the 2008–2012 period. In-between we observe the profile of a recovering market with a series of peaks that are reflected differently by the various models while the Bates one has the largest volatility of the risk measures.

5. Conclusions

This paper responds on the growing interest to measuring the financial risks caused by the new financial realities and the changed regulatory requirements both nationally and supra-nationally. Two classical measures are considered (VaR and ES) as well as one relatively novel, namely the expectile risk measure (ERM). The approach we follow is based on the assumption that the financial object follows

Table 2

Parameters for Black–Scholes, exponential tempered stable, Heston, Bates, and SV/TS models.

Parameter	BS	ETS	Heston	Bates	SV/TS
μ	-0.0252	0.0856	0.0526	0.0790	0.0885
ξ	–	–	2482.4726	161.1913	2.8933
η	–	–	0.0332	0.0248	0.0062
θ	–	–	47.4600	3.2837	0.2845
ρ	–	–	-0.2021	-0.4189	-0.6000
α_1	–	0.2154	–	–	0.0005
α_2	–	1.2755	–	–	1.3527
c_1	–	37.8024	–	–	67.7031
c_2	–	0.1425	–	–	0.0584
λ_1	–	69.1361	–	–	65.3019
λ_2	–	30.2549	–	–	14.2655
σ	0.1652	–	–	–	–
λ	–	–	–	9.5952	–
k	–	–	–	0.0022	–
δ	–	–	–	0.0332	–

a stochastic process. This way the inherent characteristics can be extracted from the historical data. The closed form formulas are derived under the assumption that only the characteristic functions of the processes are available. Also, a method for deriving the empirical risk measures based on the so called truncated expectations is provided. The standard least square errors estimator is adapted to the risk measurement by adding an logarithmic component. We consider the S&P 500 index historical sample between October 6, 2004 and July 01, 2022 – this period contains a low volatility cycles as well as the financial crises from 2008 and the recent COVID19 years. We chose five stochastic processes for these purposes, namely the geometric Brownian motion, an exponent of a tempered stable process, a stochastic volatility process with and without jumps. The jumps are presented by two additional processes – finite (compound Poisson with normally distributed innovations) and infinite activity (tempered stable). We calibrate dynamically these model using a moving sample with 1000 observations. Several empirical tests are provided to check the significance of the derived results. All of them reject the log-normal process as a good candidate – it is well known that the real observations are not normally distributed. This is due the light tails this distribution exhibits which leads to the underestimation of all risk measures. Overall, the derived results leads to the conclusion that the tempered stable jump behavior is more appropriate, and the stochastic volatility has its impact, albeit less. Also, it turns out that the correction coefficients with values $k_1 = 0.6$ (for VaR and ES) and $k_2 = 1.25$ (for VaR and ERM) guarantee smooth enough shifting between the corresponding risk measures.

To conclude, the vast application area of market risk measures determines the impact of our study. Such measures are applied to estimate the impact of market changes on positions' value, run stress tests, calculate the expected daily losses, monitor the portfolio risk profile, operationalize the critical market conditions, calculate exposures and hence the hedges, allocate capital buffers, secure transparency and prudent management, meet regulatory requirements. The research on these topics spread across two directions: comparing the advantages of risk measures (measurement accuracy, forecasting capacities, tractability and interpretability, etc.) and exploring how well the risk measure meets the regulatory requirements. Our research covers these areas.

Acknowledgments

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Appendix. Proofs of some propositions

We need the following lemma.

Lemma A.1. We have for arbitrary real v , x , and b

$$\Re \left(\frac{e^{-ixv} \Psi_{\xi}(v+ib)}{v+ib} \right) = -\Re \left(\frac{e^{ixv} \Psi_{\xi}(-v+ib)}{-v+ib} \right) \quad (\text{A.1})$$

$$\Im \left(\frac{e^{-ixv} \Psi_{\xi}(v+ib)}{v+ib} \right) = \Im \left(\frac{e^{ixv} \Psi_{\xi}(-v+ib)}{-v+ib} \right).$$

Proof. The Eqs. (A.1) follows from the fact that the pair

$$\left\{ i \frac{e^{-ixv} \Psi_{\xi}(v+ib)}{v+ib}, i \frac{e^{ixv} \Psi_{\xi}(-v+ib)}{-v+ib} \right\} \quad (\text{A.2})$$

is complex conjugate due to the same fact for the pairs $\{e^{-ixv}, e^{ixv}\}$, $\{\Psi_{\xi}(v+ib), \Psi_{\xi}(-v+ib)\}$, and $\{i(v+ib), i(-v+ib)\}$. $\square$

Together with the proofs presented below, we also refer to Kim et al. (2010) and Park (2016).

A.1. Proof of Proposition 3.1

Suppose first that $b > 0$. We have

$$\begin{aligned} F_{\xi}(x) &= \frac{1}{2\pi} \int_{-\infty}^x \int_{ib-\infty}^{ib+\infty} e^{-ivu} \Psi_{\xi}(u) du dv \\ &= \frac{1}{2\pi} \int_{ib-\infty}^{ib+\infty} \Psi_{\xi}(u) \int_{-\infty}^x e^{-ivu} dv du \\ &= \frac{i}{2\pi} \int_{ib-\infty}^{ib+\infty} \frac{e^{-ixu} \Psi_{\xi}(u)}{u} du \\ &= \frac{ie^{bx}}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-ixv} \Psi_{\xi}(v+ib)}{v+ib} dv \\ &= \frac{e^{bx}}{\pi} \int_0^{+\infty} \Im \left(\frac{e^{-ixv} \Psi_{\xi}(v+ib)}{-v+ib} \right) dv \end{aligned} \quad (\text{A.3})$$

We have used above (A) Theorem 3.1, (B) the fact that condition (3.2) allows us to interchange the integration order, (C) $\lim_{v \rightarrow -\infty} |e^{-i(\beta+ib)v}| = \lim_{v \rightarrow -\infty} e^{vb} = 0$ for every $\beta \in \mathbb{R}$ and (A.1).

Let us turn to the case $b < 0$. Having in mind $\lim_{v \rightarrow \infty} |e^{-i(\beta+ib)v}| = \lim_{v \rightarrow \infty} e^{vb} = 0$ we analogously obtain for the CCDF

$$\begin{aligned} 1 - F_{\xi}(x) &= \frac{1}{2\pi} \int_x^{+\infty} \int_{ib-\infty}^{ib+\infty} e^{-ivu} \Psi_{\xi}(u) du dv \\ &= \frac{1}{2\pi} \int_{ib-\infty}^{ib+\infty} \Psi_{\xi}(u) \int_x^{+\infty} e^{-ivu} dv du \\ &= -\frac{i}{2\pi} \int_{ib-\infty}^{ib+\infty} \frac{e^{-ixu} \Psi_{\xi}(u)}{u} du \\ &= \frac{e^{bx}}{\pi} \int_0^{+\infty} \Im \left(\frac{e^{-ixv} \Psi_{\xi}(v+ib)}{v+ib} \right) dv. \end{aligned} \quad (\text{A.4})$$

Finally, let us consider the limiting case $b = 0$. We shall use the Cauchy's integral theorem. Note that the integrated function has a singularity for $v = 0$ and hence we can choose an appropriate contour (with infinite endpoints). Let $b \in (a_1, a_2)$ and $b > 0$. We choose the region presented at Fig. 1(a). We have

$$\int_{(-\epsilon, \epsilon)} \frac{e^{-ixv} \Psi_{\xi}(v)}{v} dv = -\frac{1}{2} \oint \frac{e^{-ixv} \Psi_{\xi}(v)}{v} dv = -\pi i \quad (\text{A.5})$$

Table 3

Deviations from the real values and forecast results.

Parameter	BS	ETS	Heston	Bates	SV/TS
Averaged error	9.7756	4.3051	4.5368	4.3096	4.2394
Whole data error	7.4516	1.1083	1.1845	1.1722	1.0809
Averaged $Var(0.05)$ error	0.0018	0.0009	0.0008	0.0007	0.0007
Averaged $ES(0.05)$ error	0.0027	0.0020	0.0018	0.0047	0.0013
Averaged $ERM(0.05)$ error	0.0028	0.0009	0.0008	0.0020	0.0011
One day forecast	9446.0785	10867.8194	10836.3136	10834.1492	10861.8049
Averaged value – one day	29.5379	34.5787	34.3108	34.2134	34.6249
One weak forecast	9473.2248	10842.6018	10820.1055	10815.7427	10830.1246
Averaged value – one weak	29.5231	34.5426	34.2752	34.1824	34.5881
One month forecast	9609.9859	10839.6393	10817.9633	10816.0895	10821.3718
Averaged value – one month	29.5553	34.6076	34.3597	34.2527	34.6735

Table 4

Risk measures results for the whole data.

Parameter	Empirical	BS	ETS	Heston	Bates	SV/TS
$Var(0.05)$	0.0186	0.0173	0.0184	0.0184	0.0186	0.0186
$ES(0.05)$	0.0313/0.0188	0.0217/0.0130	0.0292/0.0175	0.0302/0.0181	0.0297/0.0178	0.0299/0.0179
$ERM(0.05)$	0.0150/0.0188	0.0121/0.0151	0.0142/0.0177	0.0147/0.0184	0.0144/0.0181	0.0145/0.0181
$Var(0.025)$	0.0261	0.0206	0.0257	0.0260	0.0257	0.0260
$ES(0.025)$	0.0407/0.0244	0.0245/0.0147	0.0368/0.0221	0.0388/0.0233	0.0377/0.0226	0.0379/0.0227
$ERM(0.025)$	0.0205/0.0257	0.0148/0.0185	0.0191/0.0239	0.0199/0.0249	0.0194/0.0243	0.0195/0.0244
$Var(0.01)$	0.0366	0.0245	0.0358	0.0373	0.0363	0.0368
$ES(0.01)$	0.0556/0.0334	0.0279/0.0167	0.0472/0.0283	0.0510/0.0306	0.0492/0.0295	0.0491/0.0294
$ERM(0.01)$	0.0290/0.0363	0.0181/0.0226	0.0262/0.0328	0.0278/0.0347	0.0269/0.0336	0.0270/0.0337

since

$$\text{Res}\left(\frac{e^{-ixv}\Psi_{\xi}(v)}{v}, 0\right) = 1. \quad (\text{A.6})$$

Using Eq. (A.3) we derive

$$\begin{aligned} F_{\xi}(x) &= \frac{i}{2\pi} \int_{ib_2-\infty}^{ib_2+\infty} \frac{e^{-ixv}\Psi_{\xi}(v)}{v} dv \\ &= \lim_{\epsilon \rightarrow 0} \frac{i}{2\pi} \left(\int_{(-\infty, -\epsilon) \cup (\epsilon, \infty)} \frac{e^{-ixv}\Psi_{\xi}(v)}{v} dv + \int_{(-\epsilon, \epsilon)} \frac{e^{-ixv}\Psi_{\xi}(v)}{v} dv \right) \\ &= \frac{1}{2} + \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{+\infty} \Im \left(\frac{e^{-ixv}\Psi_{\xi}(v)}{v} \right) dv. \end{aligned} \quad (\text{A.7})$$

Analogously, suppose that $a_1 < b < 0$ and let us consider negative contour 1(b). We have

$$\int_{(-\epsilon, \epsilon)} \frac{e^{-ixv}\Psi_{\xi}(v)}{v} dv = \pi i \quad (\text{A.8})$$

due to Eq. (A.6). Hence, using Eq. (A.4), we obtain

$$\begin{aligned} F_{\xi}(x) &= 1 + \frac{i}{2\pi} \int_{ib_2-\infty}^{ib_2+\infty} \frac{e^{-ixv}\Psi_{\xi}(v)}{v} dv \\ &= 1 + \lim_{\epsilon \rightarrow 0} \frac{i}{2\pi} \left(\int_{(-\infty, -\epsilon) \cup (\epsilon, \infty)} \frac{e^{-ixv}\Psi_{\xi}(v)}{v} dv + \int_{(-\epsilon, \epsilon)} \frac{e^{-ixv}\Psi_{\xi}(v)}{v} dv \right) \\ &= \frac{1}{2} + \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{+\infty} \Im \left(\frac{e^{-ixv}\Psi_{\xi}(v)}{v} \right) dv. \end{aligned} \quad (\text{A.9})$$

A.2. Proof of Proposition 3.2

Let us remind the following integral

$$I(v, u) := \frac{ivu + 1}{u^2} e^{-ivu} = \int v e^{-ivu} dv. \quad (\text{A.10})$$

Having in mind $\lim_{v \rightarrow -\infty} |e^{-ivu}| = 0$ and $\lim_{v \rightarrow -\infty} |I(v, u)| = 0$ when $\Im(u) = b_2 > 0$ and integral (A.10) we obtain

$$\begin{aligned} E[(\xi - x)^-] &= \frac{1}{2\pi} \int_{-\infty}^x (x - v) \int_{ib_2-\infty}^{ib_2+\infty} e^{-ivu}\Psi_{\xi}(u) du dv \\ &= \frac{1}{2\pi} \int_{ib_2-\infty}^{ib_2+\infty} \Psi_{\xi}(u) \int_{-\infty}^x (x - v) e^{-ivu} dv du \\ &= \frac{1}{2\pi} \int_{ib_2-\infty}^{ib_2+\infty} \left(\Psi_{\xi}(u) \left[ix \frac{e^{-ivu}}{u} - \frac{ivu + 1}{u^2} e^{-ivu} \right] \Big|_{v=-\infty}^{v=x} \right) du \\ &= -\frac{1}{2\pi} \int_{ib_2-\infty}^{ib_2+\infty} \frac{\Psi_{\xi}(u) e^{-ixu}}{u^2} du \\ &= \frac{e^{b_2 x}}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-ixv}\Psi_{\xi}(v + ib_2)}{(b_2 - iv)^2} dv \\ &= \frac{e^{b_2 x}}{\pi} \int_0^{+\infty} \Re \left(\frac{e^{-ixv}\Psi_{\xi}(v + ib_2)}{(b_2 - iv)^2} \right) dv \end{aligned} \quad (\text{A.11})$$

The second statement of formulas (3.7) can be obtained similarly to Kim et al. (2010), having in mind $\lim_{v \rightarrow \infty} |e^{-izv}| = 0$ and $\lim_{v \rightarrow \infty} |I(v, z)| = 0$ when $\Im(z) = b_1 < 0$.

The residual at the point zero is

$$\text{Res}\left(\frac{e^{-ixv}\Psi_{\xi}(v)}{v^2}, 0\right) = \frac{d(e^{-ixv}\Psi_{\xi}(v))}{dv} \Big|_{v=0} = \Psi'_{\xi}(0) - ix. \quad (\text{A.12})$$

Let us take the limit $b_2 \rightarrow 0$ in the first part of formula (3.8) and integrate on the contour presented at Fig. 1(a):

$$\begin{aligned} E[(\xi - x)^-] &= -\frac{1}{2\pi} \int_{ib_2-\infty}^{ib_2+\infty} \frac{e^{-ixv}\Psi_{\xi}(v)}{v^2} dv \\ &= -\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \left(\int_{(-\infty, -\epsilon) \cup (\epsilon, \infty)} \frac{e^{-ixv}\Psi_{\xi}(v)}{v^2} dv + \int_{(-\epsilon, \epsilon)} \frac{e^{-ixv}\Psi_{\xi}(v)}{v^2} dv \right) \\ &= \frac{i\Psi'_{\xi}(0) + x}{2} - \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{+\infty} \Re \left(\frac{e^{-ixv}\Psi_{\xi}(v)}{v^2} \right) dv. \end{aligned} \quad (\text{A.13})$$

We obtain analogously the positive truncated expectation using the Chauche theorem applied to the contour presented at Fig. 1(b):

$$\begin{aligned} E[(\xi - x)^+] &= -\frac{1}{2\pi} \int_{ib_1-\infty}^{ib_1+\infty} \frac{e^{-ixv}\Psi_{\xi}(v)}{v^2} dv \\ &= -\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \left(\int_{(-\infty, -\epsilon) \cup (\epsilon, \infty)} \frac{e^{-ixv}\Psi_{\xi}(v)}{v^2} dv + \int_{(-\epsilon, \epsilon)} \frac{e^{-ixv}\Psi_{\xi}(v)}{v^2} dv \right) \\ &= -\frac{i\Psi'_{\xi}(0) + x}{2} - \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{+\infty} \Re \left(\frac{e^{-ixv}\Psi_{\xi}(v)}{v^2} \right) dv. \end{aligned} \quad (A.14)$$

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